VECTOR & 3-D

VECTORS

1. VECTORS & THEIR REPRESENTATION

Vector quantities are specified by definite magnitude and definite directions. A vector is generally represented by a directed line segment, say $\overrightarrow{AB}$. A is called the **initial point** and B is called the **terminal point**. The magnitude of vector $\overrightarrow{AB}$ is expressed by $|\overrightarrow{AB}|$.

1.1 Zero Vector

A vector of zero magnitude is a zero vector i.e. which has the same initial & terminal point, is called a **Zero Vector**. It is denoted by $\vec{O}$. The direction of zero vector is indeterminate.

1.2 Unit Vector

A vector of unit magnitude in direction of a vector $\vec{a}$ is called unit vector along $\vec{a}$ and is denoted by $\hat{a}$ symbolically $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$.

1.3 Equal Vector

Two vectors are said to be equal if they have the same magnitude, direction & represent the same physical quantity.

1.4 Collinear Vector

Two vectors are said to be collinear if their directed line segments are parallel disregards to their direction. Collinear vectors are also called **Parallel Vectors** . If they have the same direction they are named as like vectors otherwise unlike vectors.

Symbolically, two non – zero vectors $\vec{a}$ and $\vec{b}$ are collinear if and only, if $\vec{a} = K\vec{b}$, where $K \in R - \{0\}$.

1.5 Coplanar Vector

A given number of vectors are called coplanar if their line segments are all parallel to the same plane. Note that **"Two Vectors Are Always Coplanar".**

1.6 Position Vector of A Point

Let O be a fixed origin, then the position vector of a point P is the vector $\overrightarrow{OP}$. If $\vec{a}$ and $\vec{b}$ are positive vectors of two points A and B, then, $\overrightarrow{AB} = \vec{b} - \vec{a}$ = pv of B – pv of A.

If $\vec{a}$ and $\vec{b}$ are the position vectors to two points A and B then the p.v. of a point which divides AB in the ratio m : n is given by : $\vec{r} = \frac{n\vec{a} + m\vec{b}}{m+n}$. **Note p.v. of mid point of AB =** $\frac{\vec{a}+\vec{b}}{2}$

2. ALGEBRA OF VECTORS

2.1 Addition of vectors

If two vectors $\vec{a}$ & $\vec{b}$ are represented by $\overrightarrow{OA}$ & $\overrightarrow{OB}$, then their sum $\vec{a}+\vec{b}$ is a vector represented by $\overrightarrow{OC}$, where OC is the diagonal of the parallelogram OACB.

- $\vec{a}+\vec{b} = \vec{b}+\vec{a}$ (commutative)
- $(\vec{a}+\vec{b})+\vec{c} = \vec{a}+(\vec{b}+\vec{c})$ (associativity)
- $\vec{a}+\vec{0} = \vec{a} = \vec{0}+\vec{a}$
- $\vec{a}+(-\vec{a}) = \vec{0} = (-\vec{a})+\vec{a}$

2.2 Multiplication of a Vector by a scalar

If $\vec{a}$ is a vector & m is a scalar, then m $\vec{a}$ is vector parallel to $\vec{a}$ whose modulus is |m| times that of $\vec{a}$. This is multiplication is called **Scalar Multiplication**. If $\vec{a}$ & $\vec{b}$ are vectors & m, n are scalars, then :

$m(\vec{a}) = (\vec{a})m = m\vec{a}$

$m(n\vec{a}) = n(m\vec{a}) = (mn)\vec{a}$

$(m+n)\vec{a} = m\vec{a} + n\vec{a}$

$m(\vec{a}+\vec{b}) = m\vec{a} + m\vec{b}$

3. TEST OF COLLINEARITY

Three points A, B, C with position vectors $\vec{a}, \vec{b}, \vec{c}$ respectively are collinear, if & only if there exist scalar x, y, z not all zero simultaneously such that; $x\vec{a} + y\vec{b} + z\vec{c} = 0$, where x +y +z =0

4. TEST OF COPLANARITY

Four points A, B, C, D with position vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ respectively are coplanar if and only if there exist scalars x, y, z w not all zero simultaneously such that $x\vec{a} + y\vec{b} + z\vec{c} + w\vec{d} = 0$ where, $x + y + z + w = 0$

5. PRODUCT OF VECTORS

5.1 Scalar product of two vectors

- $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta \; (0 \le \theta \le \pi)$

  note that if θ is acute then $\vec{a} \cdot \vec{b} > 0$ & if θ is obtuse then $\vec{a} \cdot \vec{b} < 0$

- $\vec{a} \cdot \vec{a} = |\vec{a}|^2 = \vec{a}^2, \vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$ (commutative)

  $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$ (distributive)

- $\vec{a} \cdot \vec{b} = 0 \Leftrightarrow \vec{a} \perp \vec{b}$

  $(\vec{a} \ne 0 \; \vec{b} \ne 0)$

- $(m\vec{a}) \cdot \vec{b} = \vec{a} \cdot (m\vec{b}) = m(\vec{a} \cdot \vec{b})$ (associative), where m is scalar.

- $\hat{i}.\hat{i} = \hat{j}.\hat{j} = \hat{k}.\hat{k} = 1$;

  $\hat{i}.\hat{j} = \hat{j}.\hat{k} = \hat{k}.\hat{i} = 0$

- projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$

- the angle φ between $\vec{a}$ & $\vec{b}$ is given by $\cos\phi = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$

  $0 \le \phi \le \pi$

- if $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ then

  $\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

- $|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

- $|\vec{b}| = \sqrt{b_1^2 + b_2^2 + b_3^2}$

Note...

(i) Maximum value of $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|$

(ii) Minimum values of $\vec{a} \cdot \vec{b} = -|\vec{a}||\vec{b}|$

(iii) Any vector $\vec{a}$ can be written as, $\vec{a} = (\vec{a} \cdot \hat{i})\hat{i} + (\vec{a} \cdot \hat{j})\hat{j} + (\vec{a} \cdot \hat{k})\hat{k}$.

(iv) A vector in the direction of the bisector of the angle between two vectors $\vec{a}$ & $\vec{b}$ is $\dfrac{\vec{a}}{|\vec{a}|} + \dfrac{\vec{b}}{|\vec{b}|}$.

Hence bisector of the angle between the two vectors $\vec{a}$ & $\vec{b}$ is $\lambda(\hat{a} + \hat{b})$, where $\lambda \in R^+$.

Bisector of the exterior angle between $\vec{a}$ & $\vec{b}$ is $\lambda(\hat{a} - \hat{b})$ $\lambda \in R - \{0\}$.

5.2 Vector product of two vectors

- If $\vec{a}$ & $\vec{b}$ are two vectors & θ is the angle between them then $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta \; \hat{n}$, where $\hat{n}$ is the unit vector perpendicular to both $\vec{a}$ & $\vec{b}$ such that $\vec{a}$, $\vec{b}$ & $\hat{n}$ forms a right handed screw system.

- Geometrically $|\vec{a} \times \vec{b}|$ = area of the parallelogram whose two adjacent sides are represented by $\vec{a}$ & $\vec{b}$

- $\vec{a} \times \vec{b} = \vec{0} \Leftrightarrow \vec{a}$ and $\vec{b}$ are parallel (collinear) (provided $\vec{a} \ne 0, \vec{b} \ne 0$) i.e. $\vec{a} = K\vec{b}$, where K is scalar.

- $\vec{a}\times\vec{b} \neq \vec{b}\times\vec{a}$ (not commutative)
- $(m\vec{a})\times\vec{b} = \vec{a}\times(m\vec{b}) = m(\vec{a}\times\vec{b})$ where m is scalar
- $\vec{a}\times(\vec{b}+\vec{c}) = (\vec{a}\times\vec{b})+(\vec{a}\times\vec{c})$ (distributive)
- $\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = 0$
- $\hat{i}\times\hat{j} = \hat{k}, \hat{j}\times\hat{k} = \hat{i}, \hat{k}\times\hat{i} = \hat{j}$
- If $\vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b} = b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ then

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a \\ b_1 & b_2 & b \end{vmatrix}$$

- Unit vector perpendicular to the plane of $\vec{a}$ & $\vec{b}$ is

$$\hat{n} = \pm\frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$$

- A vector of magnitude 'r' & perpendicular to the plane of $\vec{a}$ and $\vec{b}$ is $\pm\frac{r(\vec{a}\times\vec{b})}{|\vec{a}\times\vec{b}|}$
- If θ is the angle between $\vec{a}$ & $\vec{b}$ then $\sin\theta = \frac{|\vec{a}\times\vec{b}|}{|\vec{a}|\,|\vec{b}|}$
- If $\vec{a}, \vec{b}$ & $\vec{c}$ are the pv's of 3 points A, B and C then the vector area of triangle ABC $= \frac{1}{2}[\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}]$.

  The point A, B & C are collinear if $\vec{a}\times b+\vec{b}\times\vec{c}+\vec{c}\times\vec{a} = 0$
- **Area of any quadrilateral** whose diagonal vectors are $\vec{d}_1$ & $\vec{d}_2$ is given by $\frac{1}{2}|\vec{d}_1\times\vec{d}_2|$
- **Lagranges Identity** : for any two vector $\vec{a}$ & $\vec{b}$;

$$(\vec{a}\times\vec{b})^2 = |\vec{a}|^2\,|\vec{b}|^2 - (\vec{a}\,.\,\vec{b})^2 = \begin{vmatrix} \vec{a}\,.\,\vec{a} & \vec{a}\,.\,\vec{b} \\ \vec{a}\,.\,\vec{b} & \vec{b}\,.\,\vec{b} \end{vmatrix}$$

5.3 Scalar triple product

- The scalar triple product of three vectors $\vec{a}, \vec{b}$ & $\vec{c}$ is defined as :

  $\vec{a}\times\vec{b}\,.\,\vec{c} = |\vec{a}|\,|\vec{b}|\,|\vec{c}|\,\sin\theta\cos\phi$ where θ is the angle between $\vec{a}$ & $\vec{b}$ & ϕ is angle between $\vec{a}\times\vec{b}$ & $\vec{c}$

  It is also defined as $[\vec{a}\ \vec{b}\ \vec{c}]$, spelled as **box product.**
- Scalar triple product geometrically represents the volume of the parallelopied whose three coterminous edges are represented by $\vec{a}, \vec{b}$ & $\vec{c}$ i.e. $V = [\vec{a}\ \vec{b}\ \vec{c}]$
- In a scalar triple product the position of dot & cross can be interchanged i.e.

  $\vec{a}\,.\,(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\,.\,\vec{c}$ Also $[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{c}\ \vec{a}\ \vec{b}]$

  $\vec{a}\,.\,(\vec{b}\times\vec{c}) = -\vec{a}\,.\,(\vec{c}\times\vec{b})$ i.e. $[\vec{a}\ \vec{b}\ \vec{c}] = -[\vec{a}\ \vec{c}\ \vec{b}]$
- If $\vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$; $\vec{b} = b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ and

  $\vec{c} = c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$ then $[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} a_1 & a_2 & a \\ b_1 & b_2 & b \\ c_1 & c_2 & c \end{vmatrix}$.

  In general, if $\vec{a} = a_1\vec{\ell}+a_2\vec{m}+a_3\vec{n}; \vec{b} = b_1\vec{\ell}+b_2\vec{m}+b_3\vec{n}$ and

  $\vec{c} = c_1\vec{\ell}+c_2\vec{m}+c_3\vec{n}$ then $[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} a_1 & a_2 & a \\ b_1 & b_2 & b \\ c_1 & c_2 & c \end{vmatrix} [\vec{\ell}\ \vec{m}\ \vec{n}]$;

  where $\vec{\ell}, \vec{m}$ & $\vec{n}$ are non coplanar vectors.
- $\vec{a}, \vec{b}, \vec{c}$ are coplanar $\Leftrightarrow [\vec{a}\ \vec{b}\ \vec{c}] = 0$.
- Scalar product of three vectors, two of which are equal or parallel is 0.

Note...

If $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar then $[\vec{a}\ \vec{b}\ \vec{c}] > 0$ for right handed system & $[\vec{a}\ \vec{b}\ \vec{c}] < 0$ for left handed system.

- $[\hat{i}\ \ \hat{j}\ \ \hat{k}] = 1$
- $[K\ \vec{a}\ \vec{b}\ \vec{c}] = K\,[\vec{a}\ \vec{b}\ \vec{c}]$
- $[(\vec{a}+\vec{b})\ \vec{c}\ \vec{d}] = [\vec{a}\ \vec{c}\ \vec{d}] + [\vec{b}\ \vec{c}\ \vec{d}]$
- **The volume of the tetrahedron OABC** with O as origin & the pv's of A, B and C being $\vec{a}, \vec{b}\ \&\ \vec{c}$ respectively is given by $V = \frac{1}{6}[\vec{a}\ \vec{b}\ \vec{c}]$
- The position vector of the centroid of a tetrahedron if the pv's of its angular vertices are $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are given by $\frac{1}{4}[\vec{a}+\vec{b}+\vec{c}+\vec{d}]$.

Note that this is also the point of concurrrency of the lines joining the vertices to the centroids of the opposite faces and is also called the centre of the tetrahedron. In case the tetrahedron is regular it is equidistant from the vertices and the four faces of the tetrahedron.

- **Remember that :** $[\vec{a}-\vec{b}\ \ \vec{b}-\vec{c}\ \ \vec{c}-\vec{a}] = 0$ &

  $[\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}] = 2\,[\vec{a}\ \vec{b}\ \vec{c}]$

5.4 Vector triple product

Let $\vec{a}, \vec{b}, \vec{c}$ be any three vectors, then the expression $\vec{a}\times(\vec{b}\times\vec{c})$ is vector & is called vector triple product.

GEOMETRICAL INTERPRETATION OF $\vec{a}\times(\vec{b}\times\vec{c})$

Consider the expression $\vec{a}\times(\vec{b}\times\vec{c})$ which itself is a vector, since it is a cross product of two vectors $\vec{a}$ and $(\vec{b}\times\vec{c})$. Now $\vec{a}\times(\vec{b}\times\vec{c})$ is a vector perpendicular to the plane containing $\vec{a}$ and $(\vec{b}\times\vec{c})$ but $\vec{b}\times\vec{c}$ is a vector perpendicular to the plane containing $\vec{b}$ & $\vec{c}$, therefore $\vec{a}\times(\vec{b}\times\vec{c})$ is a vector lying in the plane of $\vec{b}$ & $\vec{c}$ and perpendicular to $\vec{a}$. Hence we can express $\vec{a}\times(\vec{b}\times\vec{c})$ in terms of $\vec{b}$ & $\vec{c}$ i.e. $\vec{a}\times(\vec{b}\times\vec{c}) = x\vec{b}+y\vec{c}$ where x and y are scalars.

- $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\,.\,\vec{c})\,\vec{b} - (\vec{a}\,.\,\vec{b})\,\vec{c}$

  $(\vec{a}\times\vec{b})\times\vec{c} = (\vec{a}\cdot\vec{c})\,\vec{b} - (\vec{b}\cdot\vec{c})\,\vec{a}$
- $(\vec{a}\times\vec{b})\times\vec{c} \neq \vec{a}\times(\vec{b}\times\vec{c})$

6. LINEAR COMBINATIONS

Given a finite set of vector $\vec{a}, \vec{b}, \vec{c}, ...$ then the vector $\vec{r} = x\vec{a} + y\,\vec{b} + z\vec{c} + ...$ is called a linear combination of $\vec{a}, \vec{b}, \vec{c}, ...$ for any x, y, z..... $\in$R. We have the following results :

(a) If $\vec{a}, \vec{b}$ are non zero, non-collinear vectors then

$x\vec{a} + y\vec{b} = x'\vec{a} + y'\vec{b} \Rightarrow x = x'\,;\ y = y'$

(b) **Fundamental Theorem :** Let $\vec{a}, \vec{b}$ be non zero, non collinear vectors. **Then any vector** $\vec{r}$ **coplanar with** $\vec{a}, \vec{b}$ can be expressed uniquely as a linear combination of $\vec{a}, \vec{b}$ i.e. There exist some unique x, y $\in$ R such that

$x\vec{a} + y\vec{b} = \vec{r}$

(c) If $\vec{a}, \vec{b}, \vec{c}$ are non-zero, non-coplanar vectors then :

$x\vec{a} + y\vec{b} + z\vec{c} = x'\vec{a} + y'\vec{b} + z'\vec{c} \Rightarrow x = x',\ y = y',\ z = z'$

(d) **Fundamental Theorem in Space :** Let $\vec{a}, \vec{b}, \vec{c}$ be non zero, non collinear vectors in space. **Then any vector** $\vec{r}$ can be uniquely expressed as a linear combination of $\vec{a}, \vec{b}, \vec{c}$ i.e. There exist some unique x, y, z $\in$ R such that

$x\vec{a} + y\vec{b} + z\vec{c} = \vec{r}$.

(e) If $\vec{x}_1, \vec{x}_2 .... \vec{x}_n$ are n non zero vectors & k_1, k_2,............k_n are n scalars& if the linear combination $k_1\vec{x}_1 + k_2\vec{x}_2 + ...k_n\vec{x}_n = 0 \Rightarrow k_1 = 0, k_2 = 0, ...k_n = 0$ then we say that vectors $\vec{x}_1, \vec{x}_2 .... \vec{x}_n$ are **Linearly Independent Vectors.**

(f) If $\vec{x}_1, \vec{x}_2 .... \vec{x}_n$ are not Linearly Independent then they are said to be Linearly Dependent vectors i.e. if $k_1\vec{x}_1 + k_2\vec{x}_2 + ...k_n\vec{x}_n = 0$ & if there exists at least one $k_r \neq 0$ then $\vec{x}_1, \vec{x}_2 .... \vec{x}_n$ are said to be **Linearly Dependent.**

Note...

- If $\vec{a} = 3\hat{i} + 2\hat{j} + 5\hat{k}$ then $\vec{a}$ is expressed as a Linear Combination of vectors $\hat{i}, \hat{j}, \hat{k}$. Also $\vec{a}$ $\hat{i}, \hat{j}, \hat{k}$ form a linearly dependent set of vectors. In general, **every set of four vectors is a linearly dependent system.**
- $\hat{i}, \hat{j}, \hat{k}$ are Linearly Independent set of vectors. For

  $K_1\hat{i} + K_2\hat{j} + K_3\hat{k} = 0 \Rightarrow K_1 = 0 = K_2 = K_3$
- Two vectors $\vec{a}$ & $\vec{b}$ are linearly dependent $\Rightarrow \vec{a}$ is parallel to $\vec{b}$ i.e. $\vec{a} \times \vec{b} = 0 \Rightarrow$ linear dependence of $\vec{a}$ & $\vec{b}$. Conversely if $\vec{a} \times \vec{b} \neq 0$ then $\vec{a}$ & $\vec{b}$ are linearly independent.
- **If three vectors $\vec{a}, \vec{b}, \vec{c}$ are linearly dependent, then they are coplanar i.e.** $[\vec{a}, \vec{b}, \vec{c}] = 0$ conversely, if $[\vec{a}, \vec{b}, \vec{c}] \neq 0$, then the vectors are linearly independent.

7. RECIPROCAL SYSTEM OF VECTORS

If $\vec{a}, \vec{b}, \vec{c}$ & $\vec{a}', \vec{b}', \vec{c}'$ are two sets of non-coplanar vectors such that $\vec{a}\cdot\vec{a}' = \vec{b}\cdot\vec{b}' = \vec{c}\cdot\vec{c}' = 1$ then the two systems are called Reciprocal System of vectors.

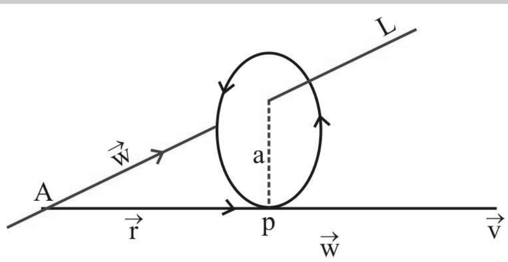
Note...

$$a' = \frac{\vec{b}\times\vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]};\ b' = \frac{\vec{c}\times\vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]};\ c' = \frac{\vec{a}\times\vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]}$$

Note...

(a) Work done against a constant force $\vec{F}$ over a displacement $\vec{s}$ is defined as $\vec{W} = \vec{F}.\vec{s}$

(b) The tangential velocity $\vec{V}$ of a body moving in a circle is given by $\vec{V} = \vec{w} \times \vec{r}$ where $\vec{r}$ is the pv of the point P.



(c) The moment of $\vec{F}$ about 'O' is defined as $\vec{M} = \vec{r} \times \vec{F}$ where $\vec{r}$ is the pv of P wrt 'O'. The direction of $\vec{M}$ is along the normal to the plane OPN such that $\vec{r}, \vec{F}$ & $\vec{M}$ form a right handed system.

(d) Moment of the couple $(\vec{r}_1 - \vec{r}_2) \times \vec{F}$ where $\vec{r}_1$ & $\vec{r}_2$ are pv's of the point of the application of the force $\vec{F}$ & $-\vec{F}$.